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On Conversion Between Amplitude and Phase Lag Parameters and Ellipse Parameters for Tidal Currents

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Contents

Conversion between the tidal current amplitude and phase lag parameters (for short, referred to as ap-parameters hereafter) and tidal current ellipse parameters (referred to as e-parameters hereafter) are not as trivial as conversion between Cartesian and polar coordinates. We spend time to figure it how to do so at one time and then forget it in half a year (or shorter, my e-folding memory scale is short, how long is yours?) later. I have just completed a tidal data assimilation project, in which I have done tons of such conversions with a sketchy MATLAB program. Recently some my colleagues inquired on how to do the conversion. Given that such inquiries are heard from time to time, I decided to pull all the relevant material together in one place for convenience in our future work. The rest of this document consists of two parts: a theory on the conversion and two streamlined programs (polished from my sketchy ones) in MATLAB (version 5 or higher).

1 Theory

1.1 Tidal ellipse and rotary components

Given tidal currents of u- (east or x-) and v- (north or y-) components, as

$$u = a_u \cos(\omega t - \phi_u) \tag{1}$$

$$v = a_v \cos(\omega t - \phi_v) \tag{2}$$

where a_u and ϕ_u are the amplitudes and phase lags for the u-components and likewise for a_v and ϕ_v, and ω is the tidal angular frequency, we can form a complex tidal current w as

$$w = u + iv \tag{3}$$

where $i = \sqrt{-1}$. If we trace w on a complex plane as time goes by a period (T=$2\pi/\omega$), we will see an ellipse. Our interest here is not only in seeing the ellipse, but more would like have the following ellipse parameters calculated:

- Maximum current velocity or semi-major axis (referred to as SEMA hereafter where appropriate).
- Eccentricity (ECC), the ratio of semi-minor to semi major axis, negative values indicating that the ellipse is traversed in a clockwise rotation;
- Inclination (INC), or angle between east (x-) and semi-major axis;

- Phase angle (PHA), i.e., the time of maximum velocity with respect to a chosen origin of time (if the phase lag is relative to Greenwich time, then the time will be also the Greenwich time).

Let us continue from eq. 3:

$$
\begin{aligned}
w &= u + iv \\
&= a_u \cos(\omega t - \phi_u) + i a_v \cos(\omega t - \phi_v) \\
&= a_u \frac{e^{i(\omega t - \phi_u)} + e^{-i(\omega t - \phi_u)}}{2} + i a_v \frac{e^{i(\omega t - \phi_v)} + e^{-i(\omega t - \phi_v)}}{2} \\
&= \frac{a_u e^{-i\phi_u} + i a_v e^{-i\phi_v}}{2} e^{i\omega t} + \frac{a_u e^{i\phi_u} + i a_v e^{i\phi_v}}{2} e^{-i\omega t} && (4) \\
&= w_p^* e^{i\omega t} + w_m e^{-i\omega t} && (5) \\
&\overset{or}{\equiv} W_p e^{i(\omega t + \theta_p)} + W_m e^{-i(\omega t - \theta_m)} && (6)
\end{aligned}
$$

where we have introduced a complex conjugate operator notation *, and

$$
w_p^* \equiv (W_p e^{-i\theta_p})^* \equiv W_p e^{i\theta_p} = \frac{a_u e^{-i\phi_u} + i a_v e^{-i\phi_v}}{2} \qquad (7)
$$

whereas w_p itself is

$$
w_p \equiv W_p e^{-i\theta_p} = \frac{a_u e^{i\phi_u} - i a_v e^{i\phi_v}}{2} \qquad (8)
$$

and

$$
w_m \equiv W_m e^{i\theta_m} = \frac{a_u e^{i\phi_u} + i a_v e^{i\phi_v}}{2} \qquad (9)
$$

where W_p, $-\theta_p$, W_m and θ_m are the amplitudes and angles of the complex variable w_p and w_m respectively, i.e.,

$$
\begin{aligned}
W_p &= \left| \frac{a_u e^{-i\phi_u} + i a_v e^{-i\phi_v}}{2} \right| && (10) \\
\theta_p &= -\tan^{-1} \left(\frac{a_u e^{-i\phi_u} + i a_v e^{-i\phi_v}}{2} \right) && (11) \\
W_m &= \left| \frac{a_u e^{i\phi_u} + i a_v e^{i\phi_v}}{2} \right| && (12) \\
\theta_m &= \tan^{-1} \left(\frac{a_u e^{i\phi_u} + i a_v e^{i\phi_v}}{2} \right). && (13)
\end{aligned}
$$

Note that we have defined the coefficient of the anti-clockwise rotating term $e^{i\omega t}$ in eq. 4 as complex conjugate of w_p instead of w_p itself. This small twist would not be needed here at all if we were only

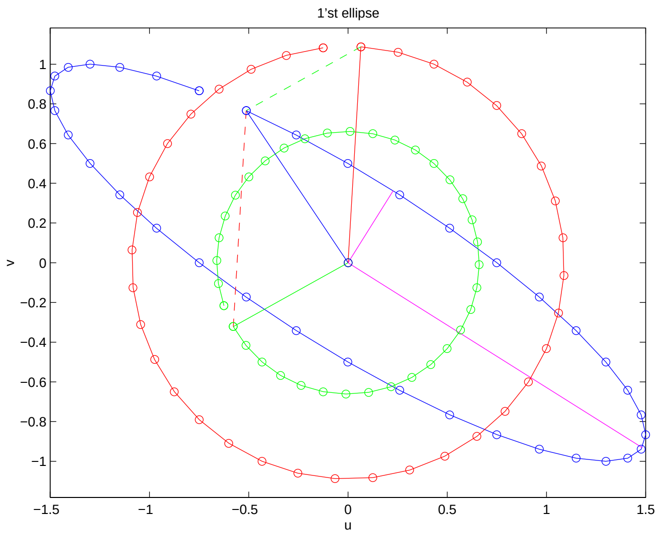

Figure 1: An ellipse can be constructed by two opposite rotating circles (red: anticlockwise circle, green: clockwise circle, blue: ellipse). The circle with a longer radius dictates the rotating direction of the ellipse.

concerned with the ap- and ep- parameters conversions. However, if we are further concerned with linear tidal dynamic equations, we will find that so defined w_p, together with w_m, allows a coupling of the two coupled momentum equations. (To decouple the momentum equations alone, one may define two complex velocities differently, but only w_p and w_m defined here can represent the two composite circles of a tidal ellipse.)

Thus, we have decomposed an ellipse into two circular components: the term with $e^{i\omega t}$ in eq. 4 or eq. 6 describes an anticlockwise circle with a radius of W_p, and the term with $e^{-i\omega t}$ describes a clockwise circle with a radius of W_m (figure 1). Depending on whether W_p is greater than, equal to or less than W_m, the ellipse will traverse either anti-clockwise, rectilinear, or clockwise.

When the two circular components are aligned in the same direction, the tidal current will reach its maximum. From, eq. 6, we can see that will happen when

$$\omega t + \theta_p \quad = \quad -\omega t + \theta_m + 2k\pi \tag{14}$$

where integer $k = 0, \pm 1, \pm 2, \pm 3, \cdots$. Denote t_{max} as a t satisfying the above criterion, then the phase angle as introduced above is ωt_{max}, which is given by

$$\text{PHA} = \omega t_{max} \quad = \quad \frac{\theta_m - \theta_p}{2} + k\pi \tag{15}$$

We need only to take its minimum value (i.e., when $k = 0$).

Substitute eq. 15 into eq. 6, we can have a current vector whose speed is maximum,

$$\begin{aligned} w_{max} &= W_p e^{i(\omega t_{max}+\theta_p)} + W_m e^{-i(\omega t_{max}-\theta_m)} \\ &= W_p e^{i\left(\frac{\theta_m+\theta_p}{2}+k\pi\right)} + W_m e^{i\left(\frac{\theta_m+\theta_p}{2}-k\pi\right)} \\ &= (W_p + W_m) e^{i\frac{\theta_m+\theta_p}{2}} \qquad\qquad \text{since } e^{ik\pi} = e^{-ik\pi} = 1 \end{aligned} \tag{16}$$

Thus, the maximum current, or semi-major axis (SEMA), is

$$\text{SEMA} = |w_{max}| = W_p + W_m \tag{17}$$

and its direction, or the inclination, is

$$\text{INC} = \tan^{-1}(w_{max}) = \frac{\theta_m + \theta_p}{2}. \tag{18}$$

When the two circular components are aligned in opposite directions, i.e.,

$$\omega t + \theta_p = -\omega t + \theta_m + (2k+1)\pi \tag{19}$$

then the tidal current reaches minimum in its speed. At this time, $t = t_{min}$

$$\omega t_{min} = \frac{\theta_m - \theta_p}{2} + (k + \frac{1}{2})\pi \tag{20}$$

and

$$\begin{aligned} w_{min} &= W_p e^{i\left(\frac{\theta_m+\theta_p}{2}+(k+\frac{1}{2})\pi\right)} + W_m e^{i\left(\frac{\theta_m+\theta_p}{2}-(k+\frac{1}{2})\pi\right)} \\ &= W_p e^{i\left(\frac{\theta_m+\theta_p}{2}\right)} e^{i\frac{\pi}{2}} + W_m e^{i\left(\frac{\theta_m+\theta_p}{2}\right)} e^{-i\frac{\pi}{2}} \\ &= \left(W_p + W_m e^{-\pi}\right) e^{i\left(\frac{\theta_m+\theta_p}{2}+\frac{\pi}{2}\right)} \\ &= \left(W_p - W_m\right) e^{i\left(\frac{\theta_m+\theta_p}{2}+\frac{\pi}{2}\right)} \end{aligned} \tag{21}$$

therefore, the minimum speed of the tidal current, or semi-minor axis (SEMI) is

$$\text{SEMI} = |w_{min}| = W_p - W_m \tag{22}$$

and its angle is

$$\tan^{-1}(w_{min}) = \frac{\theta_m + \theta_p}{2} + \frac{\pi}{2} \tag{23}$$

Thus, the eccentricity, ECC, is

$$\mathrm{ECC} = \frac{\mathrm{SEMI}}{\mathrm{SEMA}} = \frac{W_p - W_m}{W_p + W_m} \tag{24}$$

When $W_m > W_p$, ECC is negative and the ellipse rotates clockwisely.

The above is on conversion from ap-parameter to e-parameter, now let us be concerned with the other way around: given the four e-parameters of SEMA, ECC, INC, and PHA, how can we recover ap-parameters of a_u, ϕ_u, a_v and ϕ_v?

As a middle step, we need to recover W_p, θ_p, w_p, W_m, θ_m and w_m. From eqs. 17, 22 and 24, we can have

$$W_p = \frac{1+\mathrm{ECC}}{2}\mathrm{SEMA} \tag{25}$$

$$W_m = \frac{1-\mathrm{ECC}}{2}\mathrm{SEMA} \tag{26}$$

and from eqs. 15 (when $k=0$) and 18, we can have

$$\theta_p = \mathrm{INC} - \mathrm{PHA} \tag{27}$$

$$\theta_m = \mathrm{INC} + \mathrm{PHA} \tag{28}$$

Hence we can know

$$w_p = W_p e^{-i\theta_p} \tag{29}$$

$$w_m = W_m e^{i\theta_m} \tag{30}$$

We then can know further from eqs. 8 and 39 that

$$a_u e^{i\phi_u} = w_p + w_m \tag{31}$$

$$a_v e^{i\phi_v} = \frac{1}{i}\left(w_m - w_p\right) \tag{32}$$

$$\stackrel{or}{=} -i\left(w_m - w_p\right) \tag{33}$$

So,

$$a_u = \left|w_p + w_m\right|$$

$$\phi_u = \tan^{-1}\left(w_p + w_m\right) \tag{34}$$

and similarly,

$$a_v = |(w_m - w_p)| \tag{35}$$
$$\phi_v = \tan^{-1}\left(\frac{w_m - w_p}{i}\right) \tag{36}$$

1.2 Decoupling of the linear tidal momentum equations

Consider

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x} \tag{37}$$
$$\frac{\partial u}{\partial t} + fu = -g\frac{\partial \eta}{\partial y} \tag{38}$$

where all the variables are real. By adding eq. 37 and $i\times$ eq. 38, and using w defined by eq. 5, we can merge the above two equations into the following complex one,

$$\frac{\partial w}{\partial t} + ifw = -\frac{g}{2}\bigtriangledown\eta \tag{39}$$

where

$$\bigtriangledown \equiv \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}. \tag{40}$$

Assume u and v of forms of eqs. 1 similarly for η, i.e.,

$$\eta = a_\eta \cos(\omega t - \phi_\eta) \tag{41}$$

which can be split into two circular parts as we did for u and v,

$$\begin{aligned}\eta &= a_\eta \cos(\omega t - \phi_\eta)\\ &= \frac{a_\eta e^{-i\phi_\eta}}{2}e^{i\omega t} + \frac{a_\eta e^{i\phi_\eta}}{2}e^{-i\omega t}\\ &= \frac{\tilde{\eta}^*}{2} + \frac{\tilde{\eta}}{2},\end{aligned} \tag{42}$$

where

$$\tilde{\eta} = a_\eta e^{i\theta_\eta}. \tag{43}$$

Using the above equation and eqs. 5 we can rewrite eq. 39 as

$$\left[i(f+\omega)w_p^* + \frac{g}{2}\bigtriangledown\tilde{\eta}^*\right]e^{i\omega t} + \left[i(f-\omega)w_m + \frac{g}{2}\bigtriangledown\tilde{\eta}\right]e^{-i\omega t} = 0 \tag{44}$$

Since $e^{i\omega t}$ and $e^{-i\omega t}$ are linearly independent of each other, for the above eqution to hold, their coefficients must be zero, i.e.,

$$-i(f+\omega)w_p + \frac{g}{2}\,\nabla^*\,\tilde{\eta} \;=\; 0 \tag{45}$$

$$i(f-\omega)w_m + \frac{g}{2}\,\nabla\,\tilde{\eta} \;=\; 0 \tag{46}$$

where eq. 45 has been applied by an conjugate operator.

2 Programs

Two programs are included here: app2ep.m, which converts ap-parameters to e-parameters, and ep2app.m which is the inverse of app2ep.m. See the comments in the programs for more details.

2.1 app2ep.m

```
function [SEMA,  ECC, INC, PHA, w]=app2ep(Au, PHIu, Av, PHIv, plot_demo)
%
% Convert tidal amplitude and phase lag (ap-) parameters into tidal ellipse
% (e-) Parameters.
%
% Usage:
%
% [SEMA,  ECC, INC, PHA, w]=app2ep(Au, PHIu, Av, PHIv, plot_demo)
%
% where:
%
%     Au, PHIu, Av, PHIv are the amplitudes and phase lags of
%     u- and v- tidal current components. They can be vectors or
%     matrices;
%
%     plot_demo is an optional argument, when is supplied as a non-zero
%     number, the program will plot an  ellipse corresponding to
%     Au(1), PHIu(1), Av(1), and PHIv(1);
%
%     SEMA: Semi-major axes, or the maximum speed;
%     ECC:  Eccentricity, the ration of the semi-major axes over the
%           semi-minor axis;
%     INC:  Inclination, the angles  between the semi-major axes and
%           u-axis.
%     PHA:  Phase angles, the time (in angles)  when the
```

```
%          tidal currents reach their maximum speeds,  (i.e.
%          PHA=omega*tmax)
%
%          These four e-parameters will have the same dimensionality
%          (i.e., vectors, or matrices) as the input ap-parameters.
%
%     w:   Optional. If it is requested, it will be output as matrices
%          whose rows allow for plotting ellipses and whose columns are
%          for different ellipses corresponding columnwise to SEMA. For
%          example, plot(real(w(1,:)), imag(w(1,:))) will let you see
%          the first ellipse.
%
% Document:  tidal_ellipse_v2.tex and tidal_ellipse_v2.ps


  if nargin < 5
     plot_demo=0;  % by default, no plot for the ellipse
  end

% Transform the input data as column vectors
   [sizeAu_m, sizeAu_n]=size(Au); % for later transform them back.
   if sizeAu_n > 1
   Au = Au(:);
   Av = Av(:);
   PHIu = PHIu(:);
   PHIv = PHIv(:);
   end
% Assume the input phase lags are in degrees and convert them in radians.
   PHIu = PHIu/180*pi;
   PHIv = PHIv/180*pi;

% Make complex amplitudes for u and v
   i = sqrt(-1);
   u = Au.*exp(i*PHIu);
   v = Av.*exp(i*PHIv);

% Calculate complex radius of anticlockwise and clockwise circles:
   wp = (u-i*v)/2;      % for anticlockwise circles, and yes, it is (u-i*v)/2
   wm = (u+i*v)/2;      % for clockwise circles
% and their amplitudes and angles
   Wp = abs(wp);
   Wm = abs(wm);
   THETAp = -angle(wp);
```

```
   THETAm = angle(wm);

% calculate e-parameters (ellipse parameters)
    SEMA = Wp+Wm;              % Semi  Major Axis, or maximum speed
    SEMI = Wp-Wm;              % Semin Minor Axis, or minimum speed
     ECC = SEMI/SEMA;          % Eccentricity

    PHA = (THETAm-THETAp)/2;   % Phase angle, the time (in angle) when
                               % the velocity reaches the maximum
    INC = (THETAm+THETAp)/2;   % Inclination, the angle between the
                               % semi major axis and x-axis (or u-axis).

    % convert to degrees for output
    PHA = PHA/pi*180;
    INC = INC/pi*180;
    THETAp = THETAp/pi*180;
    THETAm = THETAm/pi*180;

  % flip THETAp and THETAm, PHA, and INC in the range of
  % [-pi, 0) to [pi, 2*pi), which at least is my convention.
    id = THETAp < 0;   THETAp(id) = THETAp(id)+360;
    id = THETAm < 0;   THETAm(id) = THETAm(id)+360;
    id = PHA < 0;      PHA(id) = PHA(id)+360;
    id = INC < 0;      INC(id) = INC(id)+360;

  %output e-parameter in the same matrix dimension as Au
  if sizeAu_n > 1
       SEMA = reshape(SEMA, sizeAu_m, sizeAu_n);
       SEMI = reshape(SEMI, sizeAu_m, sizeAu_n);
       PHA  = reshape(PHA,  sizeAu_m, sizeAu_n);
       INC  = reshape(INC,  sizeAu_m, sizeAu_n);
       ECC  = reshape(ECC,  sizeAu_m, sizeAu_n);
    end

  if nargout == 5 | plot_demo
     dot=pi/18;
     ot=[0:dot:2*pi-dot];
     w=wp'*exp(i*ot)+wm*exp(-i*ot);
  end


% Plot demo
  if plot_demo
```

```
  wmax = SEMA.*exp(i*INC/180*pi);
  wmin = SEMI.*exp(i*(INC/180*pi+pi/2));
%  wmax1 = wp'.*exp(i*(PHA/180*pi))     +  wm.*exp(-i*(PHA/180*pi));
%  wmin1 = wp'.*exp(i*(PHA/180*pi+pi/2))+  wm.*exp(-i*(PHA/180*pi+pi/2));
%  wmax1 and wmax should be equal to each other, so are wmin and wmin1.

figure(gcf)
clf
   n=1;   %just to plot the first ellipse for a demo.
   plot(real(w(n,:)), imag(w(n,:)))
   axis('equal');
   hold on
   plot([0 real(wmax)], [0 imag(wmax)], 'm');
   plot([0 real(wmin)], [0 imag(wmin)], 'm');
   xlabel('u')
   ylabel('v');
   title([num2str(n) '''st ellipse']
```

```
        plot(real(b(n)), imag(b(n)), 'go');
        plot(real(c(n)), imag(c(n)), 'bo');
        set(hnd_ab, 'xdata',real([a(n) a(n)+b(n)]), 'ydata', ...
        imag([a(n) a(n)+b(n)]))
        set(hnd_ba, 'xdata',real([b(n) a(n)+b(n)]), 'ydata', ...
        imag([b(n) a(n)+b(n)]))
    end


  end


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```

2.2 ep2app.m

```
function [Au, PHIu, Av, PHIv, w]=ep2app(SEMA, ECC, INC, PHA)
%
% Convert tidal ellipse parameters into amplitude and phase lag parameters.
% Its inverse is app2ep.m
%
% Zhigang Xu
% April 6, 2000
%
% Document:  tidal_ellipse_v2.tex and tidal_ellipse_v2.ps
%
   Wp = (1+ECC)/2*SEMA;
   Wm = (1-ECC)/2*SEMA;
   THETAp = INC-PHA;
```

```
   THETAm = INC+PHA;

   %convert degress into radians
   THETAp = THETAp/180*pi;
   THETAm = THETAm/180*pi;

   %Calculate wp and wm.
   wp = Wp.*exp(-i*THETAp);
   wm = Wm.*exp( i*THETAm);

   if nargout == 5
      dot = pi/18;
      ot = [0:dot:2*pi-dot];
      w = wp'.*exp(i*ot)+wm.*exp(-i*ot);
   end

   % Calculate cAu, cAv --- complex amplitude of u and v
   cAu = wp+wm;
   cAv = -i*(wm-wp);
   Au  = abs(cAu);
   Av  = abs(cAv);
   PHIu = angle(cAu)*180/pi;
   PHIv = angle(cAv)*180/pi;

   % flip angles in the range of [-180 0) to the range of [180 360).
   id = PHIu < 0; PHIu(id) = PHIu(id) + 360;
   id = PHIv < 0; PHIv(id) = PHIv(id) + 360;

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```